



TITLE:

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AUTHOR(S):

Matsuzaki, Katsuhiko

CITATION:

Matsuzaki, Katsuhiko. The action of isotropy subgroups of the modular groups on infinite dimensional Teichmuller spaces (Hyperbolic Spaces and Discrete Groups II). 数理解析研究所講究録 2002, 1270: 84-87

ISSUE DATE:

2002-06

URL:

<http://hdl.handle.net/2433/42174>

RIGHT:

The action of isotropy subgroups of the modular groups on infinite dimensional Teichmüller spaces

KATSUHIKO MATSUZAKI

松崎 克彦

Department of Mathematics, Ochanomizu University
お茶の水女子大学理学部数学科

For a compact Riemann surface R of genus greater than one, it is well known that the Teichmüller modular group (or mapping class group) $\text{Mod}(R)$ acts on the finite dimensional Teichmüller space $T(R)$ isometrically and properly discontinuously. In more details, although $\text{Mod}(R)$ has fixed points on $T(R)$, the isotropy subgroup $\text{Stab}(p)$ at any $p \in R$ is a finite group. However, this is not always the case for non-compact Riemann surfaces such as R of infinite genus or of the infinite number of punctures, for which the Teichmüller space $T(R)$ is infinite dimensional. In this case, the orbit of a point in $T(R)$ under $\text{Mod}(R)$ may be non-discrete and the isotropy subgroup $\text{Stab}(p)$ may be infinite. In this note, we consider the action of isotropy subgroups more closely. Teichmüller spaces are always assumed to be infinite dimensional hereafter.

Let R be a Riemann surface and $\text{Aut}(R)$ the group of all conformal automorphisms of R . The isotropy subgroup at the origin of the Teichmüller space $T(R)$ is identified with $\text{Aut}(R)$. Let $B(R)$ be the complex Banach space of the holomorphic quadratic differentials φ on R with the hyperbolic L^∞ -norm $\|\varphi\|$ finite. By the Bers embedding, the Teichmüller space $T(R)$ can be identified with a bounded contractible domain in $B(R)$. Then the action of $\text{Aut}(R)$ on $T(R)$ is nothing but the restriction of the action on $B(R)$ to $T(R)$, which is defined by $\varphi \mapsto g^*(\varphi) := \varphi(g) \cdot (g')^2$ for $\varphi \in B(R)$ and $g \in \text{Aut}(R)$. For a subgroup G of $\text{Aut}(R)$, we set

$$B(R/G) = \{\varphi \in B(R) \mid g^*(\varphi) = \varphi \text{ for } \forall g \in G\}.$$

This is a Banach subspace of $B(R)$, whose intersection with $T(R)$ corresponds to the Bers embedding of the Teichmüller space of the orbifold R/G .

For a subset X of $B(R)$, the limit set of X is defined as $L(X) := \overline{X} - X$. For a subgroup $G \subset \text{Aut}(R)$ and a point $\varphi \in B(R)$, the orbit of φ under G is defined as

$$G(\varphi) := \{g^*(\varphi) \in B(R) \mid g \in G\}.$$

We say that the orbit $G(\varphi)$ is discrete if it has no accumulation points in $B(R)$.

Proposition. *Let G be a subgroup of $\text{Aut}(R)$ and φ a point in $B(R)$. The orbit $G(\varphi)$ is discrete if and only if the limit set of the orbit $L(G(\varphi))$ is empty.*

Proof. If the orbit $G(\varphi)$ is discrete, then $G(\varphi)$ is closed and hence the limit set $L(G(\varphi))$ is empty. Conversely, suppose that $G(\varphi)$ is not discrete. Then there exists a sequence $\{g_n\}$ of elements in G such that $g_n^*(\varphi)$ converges to some point in $B(R)$. We may assume that this point is φ itself by replacing g_n with $g_{n+1}^{-1} \circ g_n$. Moreover, for each point $g^*(\varphi)$ in $G(\varphi)$, a sequence $\{(g \circ g_n)^*(\varphi)\} \subset G(\varphi)$ converges to $g^*(\varphi)$. If $G(\varphi)$ is closed, then this implies that $G(\varphi)$ is a closed perfect set. In a complete metric space in general, every closed perfect set contains uncountably many points. However this contradicts the fact that $G(\varphi)$ is countable. Hence $G(\varphi)$ is not closed, that is, $L(G(\varphi))$ is not empty. \square

We announce the following two results in this note. These are prototypes of our further investigation of the action of the modular groups on infinite dimensional Teichmüller spaces.

Theorem 1. *If φ belongs to the limit set $L(\cup B(R/G_n))$ for some infinite sequence of subgroups $\{G_n\}_{n=1}^\infty$ of $G = \text{Aut}(R)$, then the orbit $G(\varphi)$ is not discrete. Such an orbit always exists whenever G contains an element of infinite order.*

Proof. Take a sequence $\{\varphi_n\}$ such that $\varphi_n \in B(R/G_n)$ and $\|\varphi_n - \varphi\| \rightarrow 0$. Take an element $g_n \in G_n$ for each n and consider a sequence $\{g_n^*(\varphi)\}$. Since $g_n^*(\varphi_n) = \varphi_n$, we have

$$\begin{aligned} \|g_n^*(\varphi) - \varphi\| &= \|g_n^*(\varphi) - g_n^*(\varphi_n)\| + \|\varphi_n - \varphi\| \\ &= 2\|\varphi_n - \varphi\| \rightarrow 0, \end{aligned}$$

which means that $g_n^*(\varphi)$ converge to φ . Here $g_n^*(\varphi) \neq \varphi$ for every n because φ does not belong to any $B(R/G_n)$. Hence the orbit $G(\varphi)$ is not discrete.

Next suppose that G contains an element g of infinite order and set $G_n = \langle g^{2^{(n-1)}} \rangle$. Consider the normal covering $R/G_{n+1} \rightarrow R/G_n$ for each n . Then $G_n/G_{n+1} \cong \mathbb{Z}_2$ acts on R/G_{n+1} as the covering transformation group and thus acts on $B(R/G_{n+1})$ with the fixed point set $B(R/G_n)$. Excluding a few exceptional surfaces which do not appear in our present case, we know that the action of the Teichmüller modular group is faithful. (This was first proved in [1]. Another proof was given in [2].) This implies that the containment $B(R/G_n) \subset B(R/G_{n+1})$ is proper. Therefore we have a strictly increasing sequence of closed subspaces

$$B(R/G_1) \subsetneq B(R/G_2) \subsetneq \cdots \subsetneq B(R/G_n) \subsetneq \cdots \subset B(R).$$

Then $L(\cup B(R/G_n))$ is not empty by the Baire category theorem. \square

Theorem 2. *Suppose that the orders of the elements of $G = \text{Aut}(R)$ is uniformly bounded. If φ does not belong to the limit set $L(\cup B(R/G_n))$ for any infinite sequence of subgroups $\{G_n\}_{n=1}^\infty$ of G , then $G(\varphi)$ is discrete.*

Proof. Assume that $G(\varphi)$ is not discrete. Then there exists a sequence $\{g_n\}$ of elements in G such that $g_n^*(\varphi)$ converges to φ as in the proof of Proposition. Also we may assume that none of $\{g_n\}$ fixes φ . For $G_n = \langle g_n \rangle$, this means that φ does not belong to $\cup B(R/G_n)$. Let $k(n)$ be the order of g_n . The average of the orbit of φ under G_n is defined as

$$P_{G_n}(\varphi) := \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} (g_n^i)^*(\varphi).$$

Then $\psi_n = P_{G_n}(\varphi)$ satisfies $g_n^*(\psi_n) = \psi_n$, which means that $\psi_n \in B(R/G_n)$.

We prove that ψ_n converge to φ . The difference is estimated by

$$\begin{aligned} \|\psi_n - \varphi\| &\leq \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \|(g_n^i)^*(\varphi) - \varphi\| \\ &\leq \frac{\sum_{i=0}^{k(n)-1} i}{k(n)} \|(g_n)^*(\varphi) - \varphi\| \\ &= \frac{k(n) - 1}{2} \|(g_n)^*(\varphi) - \varphi\|. \end{aligned}$$

Since $(g_n)^*(\varphi)$ converge to φ and since $k(n)$ is uniformly bounded, we see that this converges to 0 as $n \rightarrow \infty$. This implies that φ belongs to $L(\cup B(R/G_n))$. \square

Remark 1. Concrete examples of the point φ for which the orbit $G(\varphi)$ is not discrete was given in [3]. Theorem 1 asserts that such points always exist if G has an element of infinite order.

An infinite group the orders of whose elements are bounded is known to exist as a counterexample to the Burnside problem in the group theory. Hence, due to the uniformization theorem, we can see that there exists a Riemann surface R such that $G = \text{Aut}(R)$ satisfies the assumption of Theorem 2.

The remaining case where the orders of the elements of G are finite but not bounded seems more difficult to treat.

Remark 2. In the proof of Theorem 1, we have used the fact that if a holomorphic normal covering of non-exceptional Riemann surfaces $R \rightarrow R'$ is not trivial, then the containment $B(R) \supset B(R')$ is proper. In [4], this result is extended to any covering $R \rightarrow R'$, not necessarily normal.

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OCHANOMIZU UNIVERSITY, OTSUKA 2-1-1, BUNKYO-KU, TOKYO 112-8610, JAPAN
E-mail address: matsuzak@math.ocha.ac.jp